Stopping Times

Let $\mathcal{T} \subseteq \mathbb{R}_+$ be a time axis and let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathcal{T}}, P)$ be a filtered probability model. A random variable $\tau : \Omega \to \mathcal{T} \cup \{+\infty\}$ is called a stopping time (with respect to the filtration $\mathcal{F}_t$) if

$$\forall t \in \mathcal{T} : (\tau = t) \in \mathcal{F}_t$$

This is equivalent to

$$\forall t \in \mathcal{T} : (\tau \leq t) \in \mathcal{F}_t$$

Proof.

The proof consists of two parts:

$$(\tau = t) \in \mathcal{F}_t \Rightarrow (\tau \leq t) \in \mathcal{F}_t \qquad t \in \mathcal{T} \tag{1}$$

$$(\tau = t) \in \mathcal{F}_t \Leftarrow (\tau \leq t) \in \mathcal{F}_t \qquad t \in \mathcal{T} \tag{2}$$

The following proves (1):

$$\begin{aligned}
&\forall t \in \mathcal{T} : (\tau = t) \in \mathcal{F}_t \\
\Rightarrow\ &\forall t \in \mathcal{T} : \forall s \leq t : (\tau = s) \in \mathcal{F}_s \\
\Rightarrow\ &\forall t \in \mathcal{T} : \forall s \leq t : (\tau = s) \in \mathcal{F}_t && \text{(because } \mathcal{F}_s \subseteq \mathcal{F}_t \text{ for } s \leq t) \\
\Rightarrow\ &\forall t \in \mathcal{T} : \bigcup_{s \leq t} (\tau = s) \in \mathcal{F}_t && \text{(because } \mathcal{F}_t \text{ is a } \sigma\text{-algebra)} \\
\Rightarrow\ &\forall t \in \mathcal{T} : (\tau \leq t) \in \mathcal{F}_t
\end{aligned}$$

The following proves (2):

$$\begin{aligned}
\forall t \in \mathcal{T} : (\tau \leq t) &= (\tau < t) \cup (\tau = t) \\
\underbrace{}_{\in \mathcal{F}_t} &= \underbrace{\left(\bigcup_{s<t} (\tau = s)\right)}_{=: F_t \in \mathcal{F}_t} \cup (\tau = t)
\end{aligned} \tag{3}$$

The defined F_t is an element of $\mathcal{F}_t$ because it is an union of elements which are all in $\mathcal{F}_s$ for some $s < t$ and which are therefore also elements of $\mathcal{F}_t$ because $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$. Using (3), $(\tau = t)$ can be written as:

$$\begin{aligned}
\forall t \in \mathcal{T} : (\tau = t) &= (\tau \leq t) \setminus F_t \\
&= (\tau \leq t) \cap F_t^c \\
&= \left((\tau \leq t)^c \cup F_t\right)^c \in \mathcal{F}_t
\end{aligned}$$

Because $(\tau = t)$ can be written by taking complements and unions of elements in $\mathcal{F}_t$ and because $\mathcal{F}_t$ is an σ-algebra, it follows that $(\tau = t)$ is also an element of $\mathcal{F}_t$.

It can be concluded that both definitions of a stopping time are equivalent. $\square$

Let τ_1 and τ_2 be two stopping times (with respect to the filtration $\mathcal{F}_t$). Let the random variables $\tau_{\max}$ and $\tau_{\min}$ be defined by:

$$\tau_{max}(\omega) = \max\{\tau_1(\omega), \tau_2(\omega)\} \qquad \forall \omega \in \Omega$$
$$\tau_{min}(\omega) = \min\{\tau_1(\omega), \tau_2(\omega)\} \qquad \forall \omega \in \Omega$$

These random variables are also stopping times (with respect to the same filtration $\mathcal{F}_t$).

Proof.

$$\begin{aligned}(\tau_{\max} = t) &= ((\tau_1 = t) \cap (\tau_2 \leq t)) \cup ((\tau_1 \leq t) \cap (\tau_2 = t)) \\ &= (\underbrace{(\tau_1 = t)^c}_{\in \mathcal{F}_t} \cup \underbrace{(\tau_2 \leq t)^c}_{\in \mathcal{F}_t})^c \cup (\underbrace{(\tau_1 \leq t)^c}_{\in \mathcal{F}_t} \cup \underbrace{(\tau_2 = t)^c}_{\in \mathcal{F}_t})^c\end{aligned}$$

$$\begin{aligned}(\tau_{\min} = t) &= ((\tau_1 = t) \cap (\tau_2 > t)) \cup ((\tau_1 > t) \cap (\tau_2 = t)) \cup ((\tau_1 = t) \cap (\tau_2 = t)) \\ &= ((\tau_1 = t) \cap (\tau_2 \leq t)^c) \cup ((\tau_1 \leq t)^c \cap (\tau_2 = t)) \cup ((\tau_1 = t) \cap (\tau_2 = t)) \\ &= (\underbrace{(\tau_1 = t)^c}_{\in \mathcal{F}_t} \cup \underbrace{(\tau_2 \leq t)}_{\in \mathcal{F}_t})^c \cup (\underbrace{(\tau_1 \leq t)}_{\in \mathcal{F}_t} \cup \underbrace{(\tau_2 = t)^c}_{\in \mathcal{F}_t})^c \cup (\underbrace{(\tau_1 = t)^c}_{\in \mathcal{F}_t} \cup \underbrace{(\tau_2 = t)^c}_{\in \mathcal{F}_t})^c\end{aligned}$$

Because both $(\tau_{\max} = t)$ and $(\tau_{\min} = t)$ can be written by taking complements and unions of elements in $\mathcal{F}_t$ and because $\mathcal{F}_t$ is an σ-algebra, it follows that $(\tau_{\max} = t)$ and $(\tau_{\min} = t)$ are also elements of $\mathcal{F}_t$ and are therefore also stopping times. □

If a random variable τ is a stopping time, then

$$\mathcal{F}_\tau := \{A \in \mathcal{F}; A \cap (\tau \leq t) \in \mathcal{F}_t, \forall t \geq 0\}$$

This is a σ-algebra.

Proof.

Because $\Omega \in \mathcal{F}$ and because $\Omega \in \mathcal{F}_t, \forall t \geq 0$, it follows that $\Omega \in \mathcal{F}_\tau$.

Notice that

$$F \in \mathcal{F}_\tau \Rightarrow F \cap (\tau \leq t) \in \mathcal{F}_t, \forall t \geq 0$$

Because

$$(\tau \leq t) = ((\tau \leq t) \cap F) \cup ((\tau \leq t) \cap F^c), \forall t \geq 0$$

it follows that

$$\begin{aligned}(\tau \leq t) \cap F^c &= (\tau \leq t) \cap ((\tau \leq t) \cap F)^c \\ &= (\underbrace{(\tau \leq t)^c}_{\in \mathcal{F}_t} \cup \underbrace{((\tau \leq t) \cap F)}_{\in \mathcal{F}_t})^c, \forall t \geq 0\end{aligned}$$

This implies that $(\tau \leq t) \cap F^c$ is an element of $\mathcal{F}_t$ for all $t \geq 0$, because it can be written by taking complements and unions of elements of $\mathcal{F}_t$. This is equivalent to $F^c \in \mathcal{F}_\tau$. It can be concluded that complements of elements F of $\mathcal{F}_\tau$ are also elements of $\mathcal{F}_\tau$.

The following shows that a countable union of elements of $\mathcal{F}_\tau$ is also an element of $\mathcal{F}_\tau$:

$$\begin{aligned}\forall n \in \mathbb{N} : F(n) \in \mathcal{F}_\tau &\Rightarrow \forall n \in \mathbb{N} : F(n) \cap (\tau \leq t) \in \mathcal{F}_t, \forall t \geq 0 \\ &\Rightarrow \bigcup_{n \in \mathbb{N}} [F(n) \cap (\tau \leq t)] \in \mathcal{F}_t, \forall t \geq 0 \\ &\Rightarrow \left[\bigcup_{n \in \mathbb{N}} F(n)\right] \cap (\tau \leq t) \in \mathcal{F}_t, \forall t \geq 0 \\ &\Rightarrow \bigcup_{n \in \mathbb{N}} F(n) \in \mathcal{F}_\tau\end{aligned}$$

All conditions of a σ-algebra are satisfied, so $\mathcal{F}_\tau$ is a σ-algebra. □